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LOCATION-ALLOCATION PROBLEMS

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The paper considers various location-allocation problems arising in the process of strategic planning of regional development. Thus, the problems are attractive for both commercial and state-owned companies. Process of such problems formation and development has been analyzed. Their classification depending upon demand, the number of product types, objects to be located, and type of range where their location takes place has been shown. General statement has been formulated; basic mathematical models of location-allocation problems have been demonstrated. Both methods and approaches for such problems solving have been described. Particular attention has been paid to infinite-dimensional location-allocation problems, i.e. to continuous problems of optimum set separation. In the context of the problems, the range in which objects are located, is a certain set continuously filled with consumers (or manufacturers) of specific product. Connection of optimum set separation problems with multistage location-allocation problems is considered separately in terms of two-stage problem. Mathematical model of multistage location problem being a combination of discrete location problem (as one of the stages) and a problem of optimum set separation is demonstrated. Their features and difficulties arising in the process of combined types of problem solution have been emphasized. Relevant research policy has been determined.

Keywords: location-allocation problems, optimization, problems of optimum partition of sets, multistage problems, mathematical models.

The problem setting

A great number of research efforts concern the problems of object planning and locating. Such problems are typical for practical studies as area for their location may be of various nature, structure, and characteristics and the "object" may be interpreted rather differently. Problems concerning location of different service centers (hospitals, shops, fire stations, various enterprises etc.); formation of general enterprise plans; irrigative problems; design of mobile networks are the examples of such problems. Solution of the problems involves different techniques and models depending upon available output data and conditions which in turn involve systematization of studies in progress. That is why both research and analysis of mathematical models for such problems is relevant problem.

Analysis of latter research and publications

For the first time, the problems were formulated as early as in 17th century. Their emergence and first attempts to solve them are connected with the name of Pierre Fermat who has formulated probably the first location problem currently known as Fermat point: determine the fourth point for the preset three ones in such a way that if three segments are passed into the points then sum of the three segments will be of the least value. The problem was solved partially by E. Torricelli and B. Cavalieri in 1640. As for back as in 1970 T. Simpson modified it and generalized in the context of account of arbitrary weighs and connections between objects.

From the viewpoint of object location the problem was further evolved in 1909 owing to efforts by M. Weber who used the model to determine optimum location for factories in terms of definite locations of resources and consumers. Currently the problem is known as the Weber problem being a part of general problem of geographical location of human business activities. Moreover, Weber believes that economic benefit depending upon location of factories is that very location factor (i.e. "standard factor"). In turn, he considered benefit as cuts in expenditures connected with output of products and their sales. In practice that meant the possibility to manufacture the product locally with less expenditures to compare with other places.

Currently a number of models of location problem, various optimum criteria and types of source areas as well as type of source range are available [1].

Formulation of the research objectives

Objective of the research is to analyze available models to solve location problems and identify relevant policies for their analysis.

Statement of the research basic material

It is possible to divide all location problems into the two large categories (Figure): problems of location of interrelated objects and problems of location-allocation (problems connected with location of enterprises). Category one includes problems with foregone structure of relations between objects: Weber problems, quadratic allocation problem etc. Category two does not include relations between allocated objects – "suppliers"; allocation of specific objects – "clients" between them. Such problems involve: problems on p-median and p-centres; the simplest location problem etc. Further classification of the problems is possible using various parameters (Figure), e.g. in terms of demand – with uniform or non-uniform product demand. So-called problems of optimum set separation which classification is described in [9] is one of categories of location problems with continuous range. There are also classifications relying upon the availability of restrictions for production facilities, number of enterprise types, number of sources etc. [1,2]. As for the initial information, one may talk about determined and stochastic problems, location problems under the conditions of either complete or incomplete information (location problems under fuzzy conditions) [8,9].

Give examples of several theoretical and practical problems which in the context of mathematical setting resolve into following location problems:

– scheduling;

– standardization;

– minimization of polynomials in boolean variables;

– two-level problems concerning the assortment of production selection;

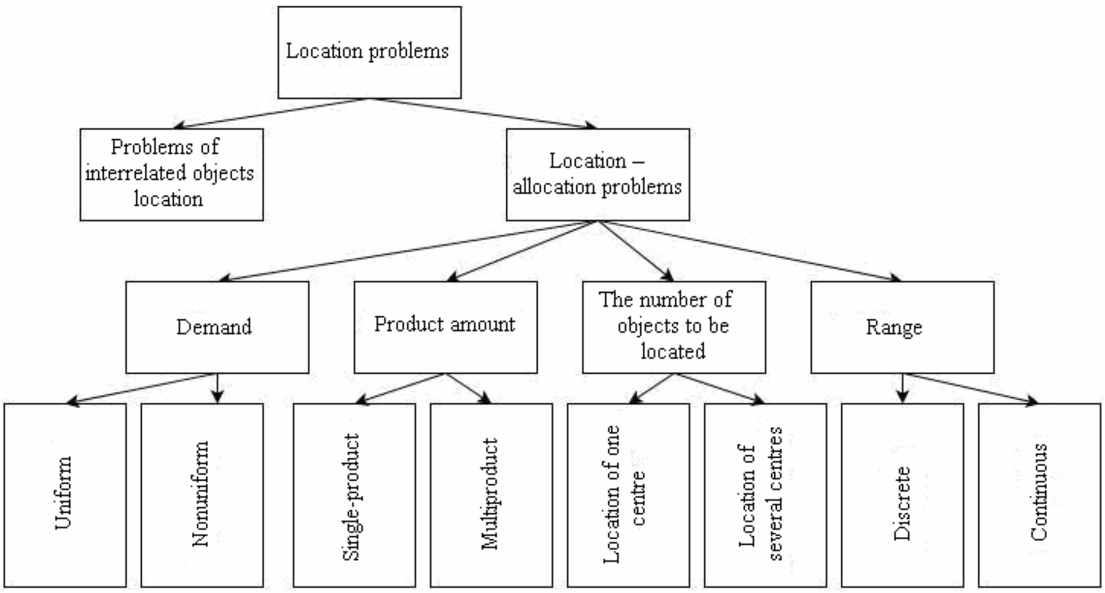
– problems concerning the development of optimum set of rows of matrix pair, optimum rows of a product and associated parts;

– multistage problems of location etc.

It should be noted that great share of the activities is based the simplest location problem plus certain conditions: for instance, placing restrictions on production volume, product types, production stages, dynamics etc.

In the context of general statement, location-allocation problem (LAP) is interpreted as follows: it is required to determine the number of new objects as well as their location coordinates; it is required to allocate transport operations between new objects and available ones. The process involves the idea that the new objects will be located in such a way to minimize transportation costs while delivering goods from objects to "consumers".

Formally, LAP is: it is required to locate N new manufacturers $(x_1,...,x_n) \in R^2$ taking into consideration of available M consumers



Types of location problems

$A = \{a_1, ..., a_m\} \in R^2$ to minimize a sum of positively weighed distances between them.

Following types of LAPs are differentiated:

– single- and multiproduct;

– with location of one or more centres (enterprises);

– single- and multisource.

Thus, mathematical statement of the simplest LAP with single source (i.e. where consumer belongs to one manufacturer) and without restriction is formulated as follows:

Problem 1. Find:

$$\min \sum_{i=1}^{N} \sum_{j=1}^{M} y_{ij} w_{ij} d(x_i, a_j)$$

if restrictions are:

$$\sum_{i=1}^{N} y_{ij} = 1, j = 1,...,M,$$

$$y_{ij} \in \{1,0\}, i = 1,...,N, j = 1,...,M.$$

In this context Boolean variables $y_{ij}, i = 1,...,N, j = 1,...,M$ involve the information that available consumer belongs to a new manufacturer; that is

$$y_{ij} = \begin{cases} 1 \text{ if } a_j \text{ belongs to } x_i, \\ 0 \text{ in any other case.} \end{cases}$$

The restriction implements a condition of the only source availability. Positive weighs $w_j, j = 1,...,M$ may mean demand of consumers a_j. In terms of specific centres the problem may be reduced to a problem of discrete set separation.

For the first time, LAP with a set of sources and without restrictions was formulated by Cooper in 1964 [2]. It looked like this:

Problem 1.1. Find

$$\min \sum_{i=1}^{N} \sum_{j=1}^{M} w_{ij} d(x_i, a_j)$$

if restrictions are:

$$\sum_{i=1}^{N} w_{ij} = r_j, j = 1,...,M,$$

$$w_{ij} \ge 0, i = 1,...,N, j = 1,...,M,$$

$$x_i \in R^2, i = 1,...,N,$$

where w_{ij} is the quantity of goods delivered from a manufacturer with i number to a consumer with j number; $d(x_i, a_j)$ is distance from manufacturer i to consumer j.

The restriction makes it possible to deliver product from r_j plants to each buyer.

Subsequently, the problems were losing their simplicity due to placing restrictions on production facilities.

$$\min \sum_{i=1}^{N} \sum_{j=1}^{M} c_{ij} w_i d(x_i, a_j)$$

if restrictions are:

$$\sum_{j=1}^{M} w_{ij} = s_i, i = 1,...,N,$$

$$\sum_{i=1}^{N} w_{ij} = r_j, j = 1,...,M,$$

$$w_{ij} \ge 0, i = 1,...,N, j = 1,...,M,$$

$$x_i \in R^2, i = 1,...,N,$$

where c_{ij} is transportation cost of a product unit per distance unit from manufacturer i to consumer j.

The restriction makes it possible to deliver product by r_j plants to each buyer as well as to restrict production facilities of s_i plants.

A number of techniques and algorithms to solve such problems are available today; among them are: branch-and-bound algorithm [3], Lagrange multipliers [4], taboo search, p-median method [5], genetic algorithm [6] and many others. However, the methods are partially continuous as there is such an assumption that the set of consumers is discrete; as a result, difficulties with consideration of demand arise. To solve the problems, the majority of current algorithms use the principle of demand aggregation. The principle involves simplification of input data set. Rather often the simplification is a result of use of arithmetic mean, mode, or median. From time to time application of the approach factors into significant errors which analysis is represented in paper [7].

It should be noted that lately more and more researchers have paid their attention to LAPs under the conditions of fuzziness. The problems differ in the consideration of the fact that in the majority of cases it is rather difficult (and sometimes impossible) to obtain truthful information concerning environmental conditions. That is why models of such problems include stochastic and fuzzy components. Generally, consumer demand is used as unidentified factor.

Currently a great number of papers involve continuity of demand. Conditionally, the problems may be divided into problems with uniform demand allocation and nonuniform one. However, as practice demonstrates, if LAPs are solved in terms of standard statements, difficulties arise when restrictions are taken into consideration.

Progress of optimum set separation (OSS) [8, 9] helped determine a method to solve infinite-dimensional location problems.

OSS problems can be conditionally divided into the two categories: discrete problems and continual problems. Discrete problems (belonging to category one) are characterized by the fact that certain finite set is subject to separation. Continual problems (belonging to category two) are characterized by availability of continual set being subject to separation. Such type of problems is relatively new and implementation is more laborious. Nevertheless, the necessity to develop algorithms for solving continual problems is indisputable as the great number of practical problems may be described with the help of such models.

Continual linear problem concerning set separation is formulated as follows [8]: let Ω be closed, restricted, observable according to Lebesgue set of Euclidean space E_n. It should be separated into N subsets $\Omega_1, \Omega_2, ..., \Omega_N$ observed according to Lebesgue; centres of the subsets? $\tau_1, \tau_2, ..., \tau_N$ should be located within range to minimize a functional:

$$F(\Omega_1, \Omega_2, ..., \Omega_N, \tau_1, \tau_2, ..., \tau_N) = \sum_{i=1}^{N} \int_{\Omega_i} c_i(x, \tau_i) \rho(x) dx$$

if restrictions are:

$$\int_{\Omega_i} \rho(x) dx \le b_i, i = \overline{1, N},$$

$$mes(\Omega_i \cap \Omega_j) = 0, i \ne j, i, j = \overline{1, N},$$

$$\sum_{i=1}^{N} \Omega_i = \Omega. .$$

Functions $c_i(x, \tau_i), i = \overline{1, N}$ are real-valued, restricted, observable according to argument x by Ω, and convex according to argument τ_i for all $i = \overline{1, N}$; $\rho(x)$ is real-valued, integrated function determined by ; are predetermined real-valued numbers satisfying the conditions of the problem solving.

Today problems of multistage production process location are claiming more and more attention to minimize total expenditures connected with delivery of product and primary material and to cover certain service area. They are another category of location-allocation problems being generalization of multistage transportation and production problems being studied actively. The problems mean that there are several groups of objects to be located. Each group has its own set of possible locations, and there is certain order of relations between them.

Two-stage problem can be used to illustrate a well-known formulation of multistage transportation and production problems [10]. Two-stage transportation and production problem is that one illustration manufacturing processes concerning one type of product, its delivery to the plants processing it into another type of product, its manufacturing and delivery to the end consumers. The simplest statements of such a problem consider two products – “raw material” and “end product”. However, the greater number of names (“raw material”, “semiproduct”, “end product”) are possible. In such a case we will talk on multistage problems. Under certain conditions, the formally multiproduct problems may be reduced to singe-product problems of a staggered type, i.e. the problems in which all the non-zero coefficients of each raw of output matrix of a simplex table have one and the same sign.

Contextual statement of a multistage transportation and production problem can be formulated as follows.

Assume that N = {1,... ,n} is a set of end product demand points, $M_r \subset N$ is a set of possible location points of r[th] stage, $1 \le r \le p$; g_i^r is expenses for location of the enterprise of r[th] stage within point r, $g_i^r \ge 0$; c_{ij} is expenses for transportation of a product unit prom point r to point j, $c_{ij} \ge 0, i, j \in N$; b_j is the demand volume in point j, $b_j > 0, j \in N$.

It is meant that each point of end product demand as well as each point of any manufacturing level is supplied with the product by one manufacturer only; in this context, enterprise of r[th] level is supplied with the product by the enterprise of $(r+1)$ [th] level, $1 \le r \le p-1$.

It is required to select subsets of location point at every level (stage) $I^r \subset M_r, r = 1, ..., p$ and implement allocation of the selected enterprises within demand points in such a way to minimize total expenditures connected with location of all the selected enterprises and product transportation.

Demonstrate mathematical statement of such allocation problem (AP) when two stages are available, and Boolean variables of selection and allocation are used respectively [10]:

Assume that $x_i = 1\,(y_k = 1)$ if the enterprise of 1st (2nd) level is located within $i \in M_1\ (k \in M_2)$ point and $x_i = 0\,(y_k = 0)$ in any other case; $x_{kij} = 1$ if jth demand point is served by kth point of 2nd level through ith point of 1st level and $x_{kij} = 0$ in any other case.

Formal statement of the simplest multistage problem is as follows:

$$\sum_{i \in M_1} g_i^1 x_i + \sum_{k \in M_2} g_k^2 y_k + \\ + \sum_{j \in N} b_j \sum_{k \in M_2} \sum_{i \in M_1} (c_{ki} + c_{ij}) x_{kij} \to \min,$$

$$\sum_{k \in M_2} \sum_{i \in M_1} x_{kij} = 1,\ j \in N,$$

$$\sum_{k \in M_2} x_{kij} \le x_i,\ j \in N, \mathrm{i} \in M_1,$$

$$\sum_{i \in M_1} x_{kij} \le y_k,\ j \in N, \mathrm{k} \in M_2,$$

$$x_i, y_k, x_{kij} \in \{0,1\}.$$

Minimization of all total expenditures for manufacturing process and transportation of raw material as well as end product is a target function.

Numerical algorithms and methods to solve such problems considering greater dimension and complexity have been developed lately. Numerous papers including recent publications represent the obtained results. In this context orientation toward the use of heuristic algorithms is observed as they do not involve complicated theoretical demonstrations. However, infinite-dimensional multistage problems did not experience any analysis due to their more complicated implementation. It should be noted that there is a great variety of ranges where similar category of problems takes place. In terms of such problems, a set is continuous by its nature and available discrete models involve simplifications which can effect end result considerably. That can be demonstrated with the help of problems where manufacturers of one (or several) stages may be located within any range point rather than being concentrated within certain points [11].

Introduce following symbols to develop mathematical model: Щ is the range within which enterprises are located; N is the required quantity of enterprises of stage 1, M is the required quantity of enterprises of stage 2; b_i is the capacity of ith enterprise of stage 1; b_j is the capacity of jth enterprise of stage 2; J is the set of possible location points of enterprises of stage 2, $J = \{\tau_1, \tau_2, ..., \tau_{M_1}\}$, $c_i^I(x, \tau_i)$ is delivery cost for raw material unit from point $x \in \Omega$ to ith enterprise of stage 1; $c_{ij}^{II} = c(\tau_i, \tau_j)$ is the cost of raw material unit delivery from ith enterprise of stage 1 to jth enterprise of stage 2; K is a set of consumers; $c_{jk}^{III} = c(\tau_j, \tau_k)$ is the cost of delivery from jth enterprise of stage 2 to kth consumer; b_k is the demand of kth consumer; $\rho(x)$ is the amount of resource within x point of Ω range; A_i^r is expenditures for ith enterprise rth stage; $\tau_i^r = (\tau_{i1}^r, \tau_{i2}^r)$ are coordinates of ith enterprise rth stage; $\tau_k = (\tau_1, \tau_2)$ are predetermined coordinates of consumer i., v_{ij}^{II} is the volume of product supplied from ith enterprise of stage 1 to jth enterprise of stage 2; v_{jk}^{III} is the volume of product supplied from jth enterprise of stage 2 to kth consumer.

Suppose:

$$\lambda_j = \begin{cases} 1 \text{ if enterprise of stage 2 is located in point j} \\ 0 \text{ in any other case} \end{cases}$$

Then mathematical model may be expressed in the form of: minimize

$$\sum_{i=1}^{N} A_i + \sum_{j=1}^{M} A_j \lambda_j + \\ + \sum_{i=1}^{N} \int_{\Omega_i} c_i^I(x, \tau_i) \rho(x) dx + \\ + \sum_{i=1}^{N} \sum_{j=1}^{M} c_{ij}^{II} v_{ij}^{II} \lambda_j + \sum_{j=1}^{M} \sum_{k=1}^{K} c_{jk}^{III} v_{jk}^{III} \lambda_j$$

if restrictions are:

$$\int_{\Omega_i} \rho(x) dx \le b_i^I,\ i = \overline{1, N},$$

$$\sum_{j=1}^{M} v_{ij}^{II} \lambda_j \le b_i^I,\ i = \overline{1, N},$$

$$\sum_{j=1}^{M} v_{ij}^{II} \lambda_j \le b_i^I \lambda_j, i = \overline{1,N}, \mathrm{j} = \overline{1,M},$$

$$\sum_{j=1}^{M} v_{ij}^{II} \lambda_j \ge b_k, i = \overline{1,N}, k = \overline{1,K},$$

$$\bigcup_{i=1}^{N} \Omega_i = \Omega, \Omega_i \cap \Omega_j = 0, i \ne j, i, j = 1,2,...,N,$$

$$v_{ij}^{II} \ge 0, v_{jk}^{III} \ge 0, i = \overline{1,N}, j = \overline{1,M}, k = \overline{1,K},$$

$$\lambda_j \in \{0,1\}, \tau^{\mathrm{I}} = (\tau_1^I, \tau_2^I, ..., \tau_N^I), \tau^{\mathrm{I}} \in \Omega^N.$$

Complexity of such problems is that mathematical model involves both discrete part and continuous one which suppose relevant combined methods of solution.

Conclusions

Nowadays studies concerning infinite-dimensional multistage location problems are almost not available due to their complexity. However, there is the whole raw of ranges where similar problems take place. In such problems outgoing set is continuous by its nature and the available discrete models need great number of simplifications effecting the end result. Thus, the development of models of infinite-dimensional multistage location problems as well as their solution methods is a rather topical task.

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ЗАДАЧІ РОЗМІЩЕННЯ-РОЗПОДІЛУ

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У статті розглянуто задачі розміщення-розподілу, які виникають при стратегічному плануванні розвитку регіону і тому представляють інтерес для комерційних та державних компаній. Проаналізовано процес формування та розвиток таких задач. Представлено їх класифікацію в залежності від: попиту, кількості видів продукції, об'єктів, що розміщуються, та виду області, в якій здійснюється розміщення. Сформульовано загальну постановку і приведені основні математичні моделі задач розміщення-розподілу. Описано методи та підходи до розв'язування таких задач. Особливу увагу приділено нескінченновимірним задачам розміщення-розподілу, а саме неперервним задачам оптимального розбиття множин. В цих задачах область, в якій проводиться розміщення об'єктів являє собою деяку множину, неперервно заповнену споживачами (або виробниками) певного виду продукції. Окремо розглянуто зв'язок задач оптимального розбиття множин із багатоетапними задачами розміщення-розподілу, на прикладі двоетапної задачі. Представлено математичну модель багатоетапної задачі розміщення, яка є комбінацією дискретної задачі розміщення (як одного з етапів) і задачі оптимального розбиття множин. Відзначено їх особливості та труднощі, які виникають в процесі розв'язування комбінованих видів задач. Виділено актуальні напрямки дослідження.

Ключові слова: задачі розміщення-розподілу, оптимізація, задачі оптимального розбиття множин, багатоетапні задачі, математичні моделі.

ЗАДАЧИ РАЗМЕЩЕНИЯ-РАСПРЕДЕЛЕНИЯ

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В статье рассмотрены задачи размещения-распределения, которые возникают при стратегическом планировании развития региона и представляют интерес для коммерческих и государственных компаний. Проанализирован процесс формирования и развития таких задач. Представлены их классификации в зависимости от: спроса, количества видов продукции, размещаемых объектов и вида области, в которой осуществляется размещение. Сформулирована общая постановка и приведены основные математические модели задач размещения-распределения. Описаны методы и подходы к решению таких задач. Особое внимание уделено бесконечномерным задачам размещения-распределения, а именно непрерывным задачам оптимального разбиения множеств. В этих задачах область, в которой проводится размещение объектов, представляет собой некоторое множество, непрерывно заполненное потребителями (или производителями) определенного вида продукции. Отдельно рассмотрена связь задач оптимального разбиения множеств с многоэтапными задачами размещения-распределения на примере двухэтапной задачи. Представлена математическая модель многоэтапной задачи размещения, которая является комбинацией дискретной задачи размещения (как одного из этапов) и задачи оптимального разбиения множеств. Отмечено их особенности и трудности, которые возникают в процессе решения комбинированных видов задач. Выделены актуальные направления исследования.

Ключевые слова: задачи размещения-распределения, оптимизация, задачи оптимального разбиения множеств, многоэтапные задачи, математические модели.